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Integrable deformations of oscillator chains from quantum algebras

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Abstract. A family of completely integrable nonlinear deformations of systems of N harmonic oscillators are constructed from the non-standard quantum deformation of the $sl(2,\mathbb{R})$ algebra. Explicit expressions for all the associated integrals of motion are given and the long-range nature of the interactions introduced by the deformation is shown to be linked to the underlying co-algebra structure. Separability and superintegrability properties of such systems are analysed, and their connection with classical angular momentum chains is used to construct a non-standard integrable deformation of the XXX hyperbolic Gaudin system.

1. Introduction

The construction of integrable systems is an outstanding application of Lie algebras in both classical and quantum mechanics [1, 2]. In fact, the very definition of integrability is based on the concept of involutivity of the conserved quantities with respect to a (either Poisson or commutator) Lie bracket. During recent years, many new results concerning 'quantum' deformations of Lie algebras and groups have been obtained (see, for instance [3]), and this work has extended in many different directions the original deformations that appeared in the context of (classical and quantum) inverse scattering methods [4]. Therefore, the question concerning whether all of these new nonlinear algebraic structures can be connected in a systematic way with the integrability properties of a certain class of dynamical systems arises as a keystone for future developments in the subject.

The aim of this paper is to answer this question in the affirmative by explicitly constructing some N-dimensional systems through the general and systematic construction of integrable systems from co-algebras that has been introduced in [5]. Such a procedure is essentially based on the role that the co-algebra structure, i.e. the existence of a homomorphism $\Delta : A \rightarrow A \otimes A$ defined on a one-particle dynamical algebra A, plays in the propagation of the integrability from the one-body problem to a general N-particle Hamiltonian with co-algebra symmetry. In this framework, quantum algebras (which are just co-algebra deformations) can be interpreted as dynamical symmetries that generate in a direct way a large class of integrable deformations. In order to extract the essential properties of the systems associated with quantum algebras, we shall concentrate here on the explicit construction and analysis of integrable deformations of (classical mechanical) oscillator chains obtained from quantum $sl(2,\mathbb{R})$ co-algebras. We recall that quantum algebra deformations of $sl(2,\mathbb{R})$ are basic in quantum algebra theory and can be found, for instance, in [3].

In the next section we briefly summarize the general construction of [5] and fix the notation. Section 3 deals with oscillator chains obtained from the non-standard quantum $sl(2,\mathbb{R})$ co-algebra [6–8] through a linear Hamiltonian of the type $\mathcal{H} = J_+ + \alpha\, J_-$. This deformation can be interpreted either as a direct algebraic implementation of a certain type of long-range interaction or, equivalently, in relation with a certain integrable perturbation of the motion of a particle under any central potential in the N-dimensional Euclidean space. Through these examples we will show an intrinsic connection between quantum deformations and nonlinear interactions depending on the momenta. Next, the construction of anharmonic chains is studied, thus showing the number of integrable systems that can be easily derived by following the present approach with different choices for the generating Hamiltonian $\mathcal{H}$.

The problem of separation of variables of these oscillator chains is analysed in section 4. As a result, it is shown that the integrable deformation introduced in section 3 is not separable. However, some other choices for $\mathcal{H}$ lead to Stäckel systems, thus preserving separability and superintegrability after deformation. In section 5 we also present the direct relationship between these $sl(2,\mathbb{R})$ oscillator chains and classical spin models, and we explicitly construct the non-standard deformation of the classical XXX hyperbolic Gaudin system [9–12]. As happened with the standard deformation, the non-standard one generates a complicated variable range [13] integrable interaction. Some final remarks and comments close the paper.

2. From co-algebras to integrable Hamiltonians

The main result of [5] can be summarized as follows: any co-algebra (A,Δ) with Casimir element C can be considered as the generating symmetry that, after choosing a non-trivial representation, gives rise to a large family of integrable systems in a systematic way. Here, we shall consider classical mechanical systems and, consequently, we shall make use of Poisson realizations D of Lie and quantum algebras of the form $D : A \to C^\infty(q,p)$. However, we recall that the formalism is also directly applicable to quantum mechanical systems.

Let (A,Δ) be a (Poisson) co-algebra with generators X_i $(i = 1,\dots,l)$ and Casimir element $C(X_1,\dots,X_l)$. Therefore, the co-product $\Delta : A \to A\otimes A$ is a Poisson map with respect to the usual Poisson bracket on $A\otimes A$:

$$\left\{X_i\otimes X_j, X_r\otimes X_s\right\}_{A\otimes A} = \{X_i,X_r\}\otimes X_jX_s + X_iX_r\otimes\{X_j,X_s\}. \tag{2.1}$$

Let us consider the Nth co-product $\Delta^{(N)}(X_i)$ of the generators

$$\Delta^{(N)} : A \to A\otimes A\otimes \dots^{N)} \otimes A \tag{2.2}$$

which is obtained (see [5]) by applying recursively the two-co-product $\Delta^{(2)} \equiv \Delta$ in the form

$$\Delta^{(N)} := (id\otimes id\otimes\dots^{N-2)}\otimes id\otimes\Delta^{(2)})\circ\Delta^{(N-1)}. \tag{2.3}$$

By taking into account that the mth co-product $(m \leqslant N)$ of the Casimir $\Delta^{(m)}(C)$ can be embedded into the tensor product of N copies of A as

$$\Delta^{(m)} : A \to \{A\otimes A\otimes\dots^{m)}\otimes A\}\otimes\{1\otimes 1\otimes\dots^{N-m)}\otimes 1\} \tag{2.4}$$

it can be shown that

$$\left\{\Delta^{(m)}(C), \Delta^{(N)}(X_i)\right\}_{A\otimes A\otimes\dots^{N)}\otimes A} = 0 \qquad i = 1,\dots,l \quad m = 2,\dots,N. \tag{2.5}$$

With this in mind it can be proven [5] that, if $\mathcal{H}$ is an *arbitrary* (smooth) function of the generators of A, the N-particle Hamiltonian defined on $A \otimes A \otimes \ldots^{N)} \otimes A$ as the Nth co-product of $\mathcal{H}$

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)) \tag{2.6}$$

fulfils

$$\left\{C^{(m)}, H^{(N)}\right\}_{A\otimes A\otimes\ldots^{N)}\otimes A} = 0 \qquad m = 2, \ldots, N \tag{2.7}$$

where the $N-1$ functions $C^{(m)}$ ($m = 2, \ldots, N$) are defined through the co-products of the Casimir C

$$C^{(m)} := \Delta^{(m)}(C(X_1, \ldots, X_l)) = C(\Delta^{(m)}(X_1), \ldots, \Delta^{(m)}(X_l)) \tag{2.8}$$

and all the integrals of motion $C^{(m)}$ are in involution

$$\left\{C^{(m)}, C^{(n)}\right\}_{A\otimes A\otimes\ldots^{N)}\otimes A} = 0 \qquad m, n = 2, \ldots, N. \tag{2.9}$$

Therefore, once a realization of A on a one-particle phase space is given, the N-particle Hamiltonian $H^{(N)}$ will be a function of N canonical pairs (q_i, p_i) and is, by construction, completely integrable with respect to the usual Poisson bracket

$$\{f, g\} = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right). \tag{2.10}$$

Furthermore, its integrals of motion will be given by the $C^{(m)}$ functions, all of them functionally independent since each of them depends on the first m pairs (q_i, p_i) of canonical coordinates.

In particular, this result can be applied to universal enveloping algebras of Lie algebras $U(g)$ [14], since they are always endowed with a natural (primitive) Hopf algebra structure of the form

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i \tag{2.11}$$

where X_i is any generator of g. Moreover, since quantum algebras are also (deformed) co-algebras (A_z, Δ_z), any function of the generators of a given quantum algebra with Casimir element C_z will provide, under a chosen deformed representation, a completely integrable Hamiltonian.

3. Oscillator chains from $sl(2, \mathbb{R})$ co-algebras

The obtention of integrable oscillator chains by using the previous approach can be achieved by selecting Poisson co-algebras (A, Δ) such that the one-dimensional harmonic oscillator Hamiltonian with angular frequency ω (and unit mass) can be written as the phase space representation D of a certain function $\mathcal{H}$ of the generators of A:

$$H = D(\mathcal{H}) = p^2 + \omega^2 q^2. \tag{3.1}$$

It is well known that $sl(2, \mathbb{R})$ can be considered as a dynamical algebra for H. Hence, when deformations of $sl(2, \mathbb{R})$ co-algebras are considered, a big class of new integrable deformations of oscillator chains can be obtained.

In particular, let us introduce the Poisson bracket analogue of the non-standard deformation of $sl(2, \mathbb{R})$ [8] given by

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \qquad \{J_3, J_-\} = -2\frac{\sinh zJ_-}{z} \qquad \{J_-, J_+\} = 4J_3 \tag{3.2}$$

where z is the deformation parameter. A Poisson co-algebra structure $(U_z(sl(2,\mathbb{R})), \Delta_z)$, is obtained by means of the following co-product:

$$\begin{aligned}\Delta_z(J_-) &= J_- \otimes 1 + 1 \otimes J_- \\ \Delta_z(J_+) &= J_+ \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_+ \\ \Delta_z(J_3) &= J_3 \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_3.\end{aligned} \tag{3.3}$$

The corresponding Casimir function reads

$$\mathcal{C}_z = J_3^2 - \frac{\sinh zJ_-}{z} J_+. \tag{3.4}$$

A one-particle phase space realization D_z of (3.2) is given by

$$\begin{aligned}\tilde{f}_-^{(1)} &= D_z(J_-) = q_1^2 \\ \tilde{f}_+^{(1)} &= D_z(J_+) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \\ \tilde{f}_3^{(1)} &= D_z(J_3) = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1\end{aligned} \tag{3.5}$$

where b_1 is a real constant that labels the representation through the Casimir: $C_z^{(1)} = D_z(\mathcal{C}_z) = -b_1$.

Let us now consider the dynamical generator

$$\mathcal{H} = J_+ + \omega^2 J_-. \tag{3.6}$$

Under (3.5), we obtain a new deformation of the oscillator Hamiltonian (3.1) including a deformed centrifugal term governed by the parameter b_1:

$$H_z^{(1)} = D_z(\mathcal{H}) = \tilde{f}_+^{(1)} + \omega^2 \tilde{f}_-^{(1)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \omega^2 q_1^2 + \frac{zb_1}{\sinh zq_1^2}. \tag{3.7}$$

Now, we follow the constructive method of section 2 and derive the two-particle phase space realization from the co-product (3.3) and two copies of the realization (3.5):

$$\begin{aligned}\tilde{f}_-^{(2)} &= (D_z \otimes D_z)(\Delta_z(J_-)) = q_1^2 + q_2^2 \\ \tilde{f}_+^{(2)} &= (D_z \otimes D_z)(\Delta_z(J_+)) \\ &= \left(\frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2}\right) \mathrm{e}^{zq_2^2} + \left(\frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2}\right) \mathrm{e}^{-zq_1^2} \\ \tilde{f}_3^{(2)} &= (D_z \otimes D_z)(\Delta_z(J_3)) = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 \,\mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 \,\mathrm{e}^{-zq_1^2}.\end{aligned} \tag{3.8}$$

It can be easily checked that these functions close again the deformed algebra (3.2) under the usual Poisson bracket $\{q_i, p_j\} = \delta_{ij}$.

By following (2.6), the two-particle Hamiltonian will be given by the realization of the co-product of $\mathcal{H}$: $H_z^{(2)} = (D_z \otimes D_z)(\Delta_z(\mathcal{H})) = \tilde{f}_+^{(2)} + \omega^2 \tilde{f}_-^{(2)}$; it reads

$$H_z^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 \,\mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} p_2^2 \,\mathrm{e}^{-zq_1^2} + \omega^2(q_1^2 + q_2^2) + \frac{zb_1}{\sinh zq_1^2} \mathrm{e}^{zq_2^2} + \frac{zb_2}{\sinh zq_2^2} \mathrm{e}^{-zq_1^2}. \tag{3.9}$$

The co-product for the Casimir, $C_z^{(2)} = (D_z \otimes D_z)(\Delta_z(\mathcal{C}_z))$, leads to the following integral of the motion for (3.9):

$$C_z^{(2)} = -\frac{\sinh zq_1^2}{zq_1^2}\frac{\sinh zq_2^2}{zq_2^2}\left(q_1 p_2 - q_2 p_1\right)^2 \mathrm{e}^{-zq_1^2}\,\mathrm{e}^{zq_2^2} - (b_1\,\mathrm{e}^{2zq_2^2} + b_2\,\mathrm{e}^{-2zq_1^2})$$
$$-\left(b_1\frac{\sinh zq_2^2}{\sinh zq_1^2} + b_2\frac{\sinh zq_1^2}{\sinh zq_2^2}\right)\mathrm{e}^{-zq_1^2}\,\mathrm{e}^{zq_2^2}. \tag{3.10}$$

The N-dimensional generalization for this system follows from the realization of the co-algebra on an N-dimensional phase space. In general, an m-dimensional phase space realization is obtained through the tensor product of m copies of (3.5) applied onto the mth deformed co-product (2.3), which is in turn induced from the two-body co-product (3.3) (see [5] for an explicit example). In our case, this construction gives

$$\tilde{f}_-^{(m)} = \sum_{i=1}^{m} q_i^2$$
$$\tilde{f}_+^{(m)} = \sum_{i=1}^{m}\left(\frac{\sinh zq_i^2}{zq_i^2}p_i^2 + \frac{zb_i}{\sinh zq_i^2}\right)\mathrm{e}^{zK_i^{(m)}(q^2)}$$
$$\tilde{f}_3^{(m)} = \sum_{i=1}^{m}\frac{\sinh zq_i^2}{zq_i^2}q_i p_i\,\mathrm{e}^{zK_i^{(m)}(q^2)} \tag{3.11}$$

where the K-functions that we will use hereafter are defined by

$$K_i^{(m)}(x) = -\sum_{k=1}^{i-1} x_k + \sum_{l=i+1}^{m} x_l \tag{3.12}$$

$$K_{ij}^{(m)}(x) = K_i^{(m)}(x) + K_j^{(m)}(x)$$
$$= -2\sum_{k=1}^{i-1} x_k - x_i + x_j + 2\sum_{l=j+1}^{m} x_l \qquad i < j. \tag{3.13}$$

From now on, any sum defined on an empty set of indices will be assumed to be zero. For instance, $K_1^{(3)}(x) = x_2 + x_3$, $K_2^{(3)}(x) = -x_1 + x_3$ and $K_3^{(3)}(x) = -x_1 - x_2$.

Consequently, the N-dimensional Hamiltonian associated with the dynamical generator (3.6) is just

$$H_z^{(N)} = \tilde{f}_+^{(N)} + \omega^2\tilde{f}_-^{(N)} = \sum_{i=1}^{N}\left(\frac{\sinh zq_i^2}{zq_i^2}p_i^2 + \frac{zb_i}{\sinh zq_i^2}\right)\mathrm{e}^{zK_i^{(N)}(q^2)} + \omega^2\sum_{i=1}^{N} q_i^2. \tag{3.14}$$

This characterizes a chain of interacting oscillators where the long-range nature of the coupling introduced by the deformation is encoded through the functions $K_i^{(N)}(q^2)$. The following $N-1$ integrals of motion are deduced from the mth coproducts of the Casimir ($m = 2, \ldots, N$):

$$C_z^{(m)} = -\sum_{i<j}^{m}\frac{\sinh zq_i^2}{zq_i^2}\frac{\sinh zq_j^2}{zq_j^2}(q_i p_j - q_j p_i)^2\mathrm{e}^{zK_{ij}^{(m)}(q^2)} - \sum_{i=1}^{m} b_i\,\mathrm{e}^{2zK_i^{(m)}(q^2)}$$
$$-\sum_{i<j}^{m}\left(b_i\frac{\sinh zq_j^2}{\sinh zq_i^2} + b_j\frac{\sinh zq_i^2}{\sinh zq_j^2}\right)\mathrm{e}^{zK_{ij}^{(m)}(q^2)}. \tag{3.15}$$

We point out that the following property is useful in the previous computations:

$$\frac{\sinh(z\sum_{i=1}^{m}x_i)}{z}=\sum_{i=1}^{m}\frac{\sinh zx_i}{z}\mathrm{e}^{zK_i^{(m)}(x)}. \tag{3.16}$$

The undeformed counterpart of the above systems can be directly obtained by applying the limit $z\to 0$ in all the expressions that we have just deduced. In particular, the Poisson co-algebra $(U(sl(2,\mathbb{R})),\Delta)$ is defined by the Lie–Poisson algebra

$$\{J_3,J_+\}=2J_+\qquad\{J_3,J_-\}=-2J_-\qquad\{J_-,J_+\}=4J_3 \tag{3.17}$$

together with the primitive co-product (2.11), and the Casimir is $\mathcal{C}=J_3^2-J_-J_+$. Once the limit $z\to 0$ is computed, the deformed phase space realization (3.11), Hamiltonian (3.14) and integrals of motion (3.15) reduce to

$$f_-^{(m)}=\sum_{i=1}^{m}q_i^2\qquad f_+^{(m)}=\sum_{i=1}^{m}\left(p_i^2+\frac{b_i}{q_i^2}\right)\qquad f_3^{(m)}=\sum_{i=1}^{m}q_ip_i \tag{3.18}$$

$$H^{(N)}=\sum_{i=1}^{N}\left(p_i^2+\omega^2q_i^2+\frac{b_i}{q_i^2}\right) \tag{3.19}$$

$$C^{(m)}=-\sum_{i<j}^{m}(q_ip_j-q_jp_i)^2-\sum_{i<j}^{m}\left(b_i\frac{q_j^2}{q_i^2}+b_j\frac{q_i^2}{q_j^2}\right)-\sum_{i=1}^{m}b_i. \tag{3.20}$$

Consequently, the non-deformed Poisson co-algebra $(U(sl(2,\mathbb{R})),\Delta)$ provides an uncoupled chain of N harmonic oscillators (3.19) (all of them with the same frequency) with centrifugal terms. We remark that the (well known [1]) complete integrability of $H^{(N)}$ is obtained directly from its underlying co-algebra symmetry. Moreover, if the centrifugal terms disappear ($b_i=0$), the integrals $C^{(m)}$ are just the quadratic Casimirs of the $so(m)$ algebras with $m=2,\ldots,N$. It is also a classical result that the Hamiltonian (3.19) is $so(N)$ invariant, since it can be interpreted as the one for a particle moving on the N-dimensional Euclidean space under the central potential ω^2r^2. We stress that all these known considerations are deduced in a straightforward way from the co-algebra symmetry of the model.

3.1. A class of integrable anharmonic chains

It is also possible to consider the non-deformed Poisson co-algebra $(U(sl(2,\mathbb{R})),\Delta)$ and a more general dynamical Hamiltonian than (3.6) of the form

$$\mathcal{H}=J_++\mathcal{F}(J_-) \tag{3.21}$$

where $\mathcal{F}(J_-)$ is an arbitrary smooth function of J_-. The formalism ensures that the corresponding system constructed from (3.21) is also completely integrable, since $\mathcal{H}$ could be *any function* of the co-algebra generators. Explicitly, this means that any N-particle Hamiltonian of the form

$$H^{(N)}=f_+^{(N)}+\mathcal{F}(f_-^{(N)})=\sum_{i=1}^{N}\left(p_i^2+\frac{b_i}{q_i^2}\right)+\mathcal{F}\left(\sum_{i=1}^{N}q_i^2\right) \tag{3.22}$$

is completely integrable, and (3.20) are its integrals of motion. Obviously, in the case where $b_i=0$, this is a well known result, since (3.22) is just the Hamiltonian describing the motion of a particle in an N-dimensional Euclidean space under the action of a central potential. The linear function $\mathcal{F}(J_-)=\omega^2J_-$ leads to the previous harmonic case, and the quadratic one

$\mathcal{F}(J_-) = J_-^2$ would give us an interacting chain of quartic oscillators. Further definitions of the function $\mathcal{F}$ would give rise to many other anharmonic chains, all of them sharing the same dynamical symmetry and the same integrals of the motion.

Moreover, the corresponding integrable deformation of (3.22) is provided by a realization of (3.21) in terms of (3.11):

$$H_z^{(N)} = \sum_{i=1}^{m} \left(\frac{\sinh zq_i^2}{zq_i^2} p_i^2 + \frac{zb_i}{\sinh zq_i^2} \right) \mathrm{e}^{zK_i^{(m)}(q^2)} + \mathcal{F}\left(\sum_{i=1}^{N} q_i^2 \right) \tag{3.23}$$

and (3.15) are again the associated integrals. This example shows clearly the number of different systems that can be obtained through the same co-algebra, and the need for a careful inspection of known integrable systems in order to investigate their possible co-algebra symmetries.

4. Separation of variables and superintegrability

It is clear that $H^{(N)}$ (3.19) is the Hamiltonian of a Liouville system [2], so that we find another set of integrals of motion in involution given by

$$I_i = p_i^2 + \omega^2 q_i^2 + \frac{b_i}{q_i^2} - \frac{H^{(N)}}{N} \qquad i = 1, \ldots, N. \tag{4.1}$$

Amongst these quantities, only $N-1$ are functionally independent ($\sum_{i=1}^{N} I_i = 0$) and, obviously, the following Hamilton–Jacobi equation admits a separable solution:

$$\begin{aligned} &H^{(N)}(q_1, \ldots, q_N; p_1, \ldots, p_N) = E \qquad p_i = \frac{\partial W}{\partial q_i} \\ &W \equiv W(q_1, \ldots, q_N) = \sum_{i=1}^{N} W_i(q_i). \end{aligned} \tag{4.2}$$

The integrals of motion I_i are independent with respect to the $C^{(m)}$ (3.20) and, in general, $\{C^{(m)}, I_i\} \neq 0$. Hence, $H^{(N)}$ is a superintegrable system.

Unlike $H^{(N)}$, the deformed Hamiltonian $H_z^{(N)}$ (3.14) no longer defines a Liouville system. In order to analyse whether $H_z^{(N)}$ admits separation of variables we recall the general criterion for the separability problem of the Hamilton–Jacobi equation (4.2): this equation is separable if the N-particle Hamiltonian H verifies the following set of $N(N-1)/2$ equations [2]:

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} - \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0 \tag{4.3}$$

where $i, j = 1, \ldots, N$ and $i < j$.

If we consider the two-particle Hamiltonian $H_z^{(2)}$ (3.9), it can be checked that the single equation (4.3) is not satisfied. This is due to the long-range nature of the deformation characterized by the K-functions: if we make $K_i^{(2)} = 0$ then the general criterion of separability is fulfilled (the same happens in higher dimensions).

However, the co-algebra construction allows for an infinite family of completely integrable deformations, all of them sharing the same integrals of motion. With this in mind, it is natural to study whether some 'modifications' of the initial dynamical generator $\mathcal{H}$ (3.6) (whose non-deformed limit ($z \to 0$) lead again to $H^{(N)}$ (3.19)) enable us to find a separable deformed

Hamiltonian. As a first step we analyse the two-particle case (3.8) and consider two new 'candidates' for $\mathcal{H}$:

$$\mathcal{H}_1 = J_+ \mathrm{e}^{\alpha_1 z J_-} + \omega^2 J_- \mathrm{e}^{\beta_1 z J_-} \quad \Rightarrow \quad H_{1,z}^{(2)} = \tilde{f}_+^{(2)} \mathrm{e}^{\alpha_1 z \tilde{f}_-^{(2)}} + \omega^2 \tilde{f}_-^{(2)} \mathrm{e}^{\beta_1 z \tilde{f}_-^{(2)}} \tag{4.4}$$

$$\mathcal{H}_2 = J_+ \mathrm{e}^{\alpha_2 z J_-} + \omega^2 \left(\frac{\mathrm{e}^{\beta_2 z J_-} - 1}{\beta_2 z} \right) \quad \Rightarrow \quad H_{2,z}^{(2)} = \tilde{f}_+^{(2)} \mathrm{e}^{\alpha_2 z \tilde{f}_-^{(2)}} + \omega^2 \left(\frac{\mathrm{e}^{\beta_2 z \tilde{f}_-^{(2)}} - 1}{\beta_2 z} \right) \tag{4.5}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are real constants; note that under the limit $z \to 0$ we recover, in both cases, the non-deformed Hamiltonian $H^{(2)}$. If we impose equation (4.3) to be fulfilled, then we find two solutions for each new Hamiltonian:

$$H_{1,z}^{(2)} : \text{(i)} \quad \alpha_1 = 1 \quad \omega = 0 \qquad \text{(ii)} \quad \alpha_1 = -1 \quad \omega = 0 \tag{4.6}$$

$$H_{2,z}^{(2)} : \text{(i)} \quad \alpha_2 = 1 \quad \beta_2 = 2 \qquad \text{(ii)} \quad \alpha_2 = -1 \quad \beta_2 = -2. \tag{4.7}$$

Since the two solutions associated with $H_{1,z}^{(2)}$ arise as particular cases of those corresponding to $H_{2,z}^{(2)}$ once the frequency ω vanishes, we only consider the latter. We stress that the two solutions (4.7) do not only provide separable Hamiltonian systems in the Hamilton–Jacobi equation (4.2), they are also Stäckel systems [2]. Furthermore, this property can be generalized to the arbitrary N-particle case. In the following we construct the Stäckel description for the first solution of (4.7).

The dynamical generator we start with is given by

$$\mathcal{H} = J_+ \mathrm{e}^{z J_-} + \omega^2 \left(\frac{\mathrm{e}^{2 z J_-} - 1}{2z} \right). \tag{4.8}$$

By introducing the realization (3.11) we obtain the Hamiltonian

$$\begin{aligned} H_z^{(N)} = \sum_{i=1}^{N} \frac{\sinh z q_i^2}{z q_i^2} \mathrm{e}^{z q_i^2} \exp\left\{ 2z \sum_{k=i+1}^{N} q_k^2 \right\} \left(p_i^2 + b_i \left(\frac{z q_i}{\sinh z q_i^2} \right)^2 \right) \\ + \omega^2 \left(\frac{\exp\left\{ 2z \sum_{j=1}^{N} q_j^2 \right\} - 1}{2z} \right) \end{aligned} \tag{4.9}$$

which has the form of a Stäckel system

$$H_z^{(N)} = \sum_{i=1}^{N} a_i(q_1, \ldots, q_N) \left(\tfrac{1}{2} p_i^2 + U_i(q_i) \right) \tag{4.10}$$

provided that

$$\begin{aligned} a_i(q_1, \ldots, q_N) &= 2 \frac{\sinh z q_i^2}{z q_i^2} \mathrm{e}^{z q_i^2} \exp\left\{ 2z \sum_{k=i+1}^{N} q_k^2 \right\} \qquad i = 1, \ldots, N \\ U_1(q_1) &= \frac{b_1}{2} \left(\frac{z q_1}{\sinh z q_1^2} \right)^2 + \frac{\omega^2}{4z} \mathrm{e}^{z q_1^2} \frac{z q_1^2}{\sinh z q_1^2} \\ U_i(q_i) &= \frac{b_i}{2} \left(\frac{z q_i}{\sinh z q_i^2} \right)^2 \qquad i = 2, \ldots, N-1 \\ U_N(q_N) &= \frac{b_N}{2} \left(\frac{z q_N}{\sinh z q_N^2} \right)^2 - \frac{\omega^2}{4z} \mathrm{e}^{-z q_N^2} \frac{z q_N^2}{\sinh z q_N^2}. \end{aligned} \tag{4.11}$$

We recall that Stäckel's theorem [2] claims that a system with a Hamiltonian of the form (4.10) admits separation of variables in the Hamilton–Jacobi equation (4.2) if and only if there exists an $N \times N$ matrix B whose entries b_{ij} depend only on q_j, and such that

$$\det B \neq 0 \qquad \sum_{j=1}^{N} b_{ij}(q_j)\, a_j(q_1, \ldots, q_N) = \delta_{i1}. \tag{4.12}$$

These requirements are satisfied by our Hamiltonian (4.9); the non-zero elements of B and its determinant are found to be

$$\begin{aligned} b_{1N}(q_N) &= \frac{zq_N^2}{2\sinh zq_N^2}\,\mathrm{e}^{-zq_N^2} \qquad b_{i\,i-1}(q_{i-1}) = \frac{zq_{i-1}^2}{\sinh zq_{i-1}^2}\,\mathrm{e}^{-zq_{i-1}^2} \\ b_{ii}(q_i) &= -\frac{zq_i^2}{\sinh zq_i^2}\,\mathrm{e}^{zq_i^2} \qquad i = 2, \ldots, N \end{aligned} \tag{4.13}$$

$$\det B = \frac{1}{2}\prod_{i=1}^{N} \frac{zq_i^2}{\sinh zq_i^2}\,\mathrm{e}^{-zq_i^2}. \tag{4.14}$$

Stäckel's theorem gives N functionally independent integrals of motion in involution which have the form

$$I_j = \sum_{i=1}^{N} a_{ij}\left(\tfrac{1}{2}p_i^2 + U_i(q_i)\right) \qquad j = 1, \ldots, N \tag{4.15}$$

where a_{ij} are the elements of the inverse matrix to B. Then $a_{i1} = a_i$, so that the first integral I_1 is just the Hamiltonian. In our case, the non-zero functions a_{ij} read

$$\begin{aligned} a_{i1} &= 2\,\frac{\sinh zq_i^2}{zq_i^2}\,\mathrm{e}^{zq_i^2}\,\exp\left\{2z\sum_{k=i+1}^{N} q_k^2\right\} \qquad i = 1, \ldots, N \\ a_{ij} &= \frac{\sinh zq_i^2}{zq_i^2}\,\mathrm{e}^{zq_i^2}\,\exp\left\{2z\sum_{k=i+1}^{j-1} q_k^2\right\} \qquad i = 1, \ldots, N \quad i < j. \end{aligned} \tag{4.16}$$

Consequently, we have proven that besides the $N-1$ integrals of motion $C_z^{(m)}$ (3.15), the Hamiltonian (4.9) has another set of $N-1$ conserved quantities given by (4.15):

$$I_j = \sum_{i=1}^{j-1} \frac{\sinh zq_i^2}{2zq_i^2}\,\mathrm{e}^{zq_i^2}\,\exp\left\{2z\sum_{k=i+1}^{j-1} q_k^2\right\}\left(p_i^2 + b_i\left(\frac{zq_i}{\sinh zq_i^2}\right)^2\right) + \frac{\omega^2}{4z}\,\exp\left\{2z\sum_{k=1}^{j-1} q_k^2\right\} \tag{4.17}$$

with $j = 2, \ldots, N$. Note that the non-deformed limit for I_j has to be computed as $\lim_{z\to 0}(I_j - \omega^2/4z)$ in order to avoid divergences.

The integrals of motion I_j are functionally independent with respect to the $C_z^{(m)}$ (3.15) and, in general, $\{C_z^{(m)}, I_j\} \neq 0$. Hence we conclude that the Hamiltonian (4.9) is superintegrable.

Finally, we remark that a similar procedure can be carried out for the second solution (4.7), thus obtaining another superintegrable Hamiltonian.

5. Angular momentum chains

The connection between the $sl(2,\mathbb{R})$ oscillator chains without centrifugal terms ($b_i = 0$) and 'classical spin' systems can be also extracted from the underlying co-algebra structure.

As was shown in [5], if we substitute the canonical realizations used until now in terms of angular momentum realizations of the same abstract $sl(2,\mathbb{R})$ Poisson co-algebra, the very same construction will lead us to a 'classical spin chain' of the XXX Gaudin type on which the non-standard quantum deformation can be easily implemented.

In particular, let us consider the S realization of the $sl(2,\mathbb{R})$ Poisson algebra (3.17) given by

$$g_3^{(1)} = S(J_3) = \sigma_3^1 \qquad g_+^{(1)} = S(J_+) = \sigma_+^1 \qquad g_-^{(1)} = S(J_-) = \sigma_-^1 \tag{5.1}$$

where the classical angular momentum variables σ_l^1 fulfil

$$\{\sigma_3^1, \sigma_+^1\} = 2\sigma_+^1 \qquad \{\sigma_3^1, \sigma_-^1\} = -2\sigma_-^1 \qquad \{\sigma_-^1, \sigma_+^1\} = 4\sigma_3^1 \tag{5.2}$$

and are constrained by a given constant value of the Casimir function of $sl(2,\mathbb{R})$ in the form $c_1 = (\sigma_3^1)^2 - \sigma_-^1\sigma_+^1$.

As usual, m different copies of (5.1) (that, in principle, could have different values c_i of the Casimir) are distinguished with the aid of a superscript σ_l^i. Then, the mth order of the co-product (2.11) provides the following realization of the non-deformed $sl(2,\mathbb{R})$ Poisson co-algebra:

$$g_l^{(m)} = (S \otimes \ldots^{m)} \otimes S)(\Delta^{(m)}(\sigma_l)) = \sum_{i=1}^{m} \sigma_l^i \qquad l = +, -, 3. \tag{5.3}$$

Now, we apply the usual construction and take $\mathcal{H}$ from (3.6). Consequently, the uncoupled oscillator chain (3.19) with all $b_i = 0$ is equivalent to the Hamiltonian

$$H^{(N)} = g_+^{(N)} + \omega^2 g_-^{(N)} = \sum_{i=1}^{m} (\sigma_+^i + \omega^2 \sigma_-^i) \tag{5.4}$$

and the Casimirs $C^{(m)}$ read ($m = 2, \ldots, N$)

$$C^{(m)} = (g_3^{(m)})^2 - g_-^{(m)} g_+^{(m)} = \sum_{i=1}^{m} c_i + \sum_{i<j}^{m} (\sigma_3^i\sigma_3^j - \sigma_-^i\sigma_+^j - \sigma_-^j\sigma_+^i). \tag{5.5}$$

Note that (5.5) are (up to constants) the classical angular momentum analogues of XXX Gaudin Hamiltonians of the hyperbolic type [9–11]. In other words, if we consider the $sl(2,\mathbb{R})$ Casimir function as the dynamical Hamiltonian $\mathcal{H} = J_3^2 - J_- J_+$, such a Gaudin system can be obtained through the co-algebra symmetry [5, 14]. As a consequence, a non-standard deformation of the Gaudin system can be now constructed through the deformed Casimir by taking

$$\mathcal{H} = J_3^2 - \frac{\sinh zJ_-}{z} J_+. \tag{5.6}$$

The deformed angular momentum realization corresponding to $U_z(sl(2,\mathbb{R}))$ is

$$\begin{aligned} &\tilde{g}_-^{(1)} = S_z(J_-) = \sigma_-^1 \qquad \tilde{g}_+^{(1)} = S_z(J_+) = \frac{\sinh z\sigma_-^1}{z\sigma_-^1}\sigma_+^1 \\ &\tilde{g}_3^{(1)} = S_z(J_3) = \frac{\sinh z\sigma_-^1}{z\sigma_-^1}\sigma_3^1 \end{aligned} \tag{5.7}$$

where the classical coordinates σ_l^1 are defined on the cone $c_1 = (\sigma_3^1)^2 - \sigma_-^1\sigma_+^1 = 0$, that is, we are considering the zero realization. It is easy to check that the mth order of the co-product (3.3) realized in the above representation (5.7) leads to the following functions:

$$\begin{aligned} \tilde{g}_-^{(m)} &= \sum_{i=1}^{m} \sigma_-^i \qquad \tilde{g}_+^{(m)} = \sum_{i=1}^{m} \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_+^i \mathrm{e}^{zK_i^{(m)}(\sigma_-)} \\ \tilde{g}_3^{(m)} &= \sum_{i=1}^{m} \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \sigma_3^i \mathrm{e}^{zK_i^{(m)}(\sigma_-)} \end{aligned} \tag{5.8}$$

that define the non-standard deformation of (5.3). Therefore, the Nth co-product of the deformed Casimir gives rise to the non-standard Gaudin system

$$\begin{aligned} H_z^{(N)} \equiv C_z^{(N)} &= (\tilde{g}_3^{(N)})^2 - \frac{\sinh z\tilde{g}_-^{(N)}}{z}\, \tilde{g}_+^{(N)} \\ &= \sum_{i<j}^{N} \frac{\sinh z\sigma_-^i}{z\sigma_-^i} \frac{\sinh z\sigma_-^j}{z\sigma_-^j}\, \mathrm{e}^{zK_{ij}^{(N)}(\sigma_-)} (\sigma_3^i\sigma_3^j - \sigma_-^i\sigma_+^j - \sigma_+^i\sigma_-^j) \end{aligned} \tag{5.9}$$

that, by construction, commutes with all lower-dimensional Gaudin Hamiltonians $C_z^{(m)}$ with $m < N$ and also with any of the N-site representations (5.8) of the generators of the deformed algebra. This Hamiltonian is the angular momentum counterpart to (3.15) for $b_i = 0$.

If we recall the result coming from the standard deformation of $sl(2, \mathbb{R})$ [5]

$$C_z^{(N)} = 2 \sum_{i<j}^{N} \frac{\sinh(\frac{1}{2}z\bar{\sigma}_3^i)}{\bar{\sigma}_3^i\, z/2} \frac{\sinh(\frac{1}{2}z\bar{\sigma}_3^j)}{\bar{\sigma}_3^j\, z/2}\, \mathrm{e}^{z\, K_{ij}^{(N)}(\bar{\sigma}_3)/2} \big(\bar{\sigma}_3^i\, \bar{\sigma}_3^j - \bar{\sigma}_1^i\, \bar{\sigma}_1^j - \bar{\sigma}_2^i\, \bar{\sigma}_2^j\big) \tag{5.10}$$

where $(\bar{\sigma}_3^i)^2 - (\bar{\sigma}_1^i)^2 - (\bar{\sigma}_2^i)^2 = 0$, we observe a strong formal similarity with respect to (5.9) since the deformation can be interpreted in both cases as the introduction of a variable range interaction in the model (compare (5.9) and (5.10) with (5.5)). However, within the non-standard deformation the variable range factor is constructed in terms of functions of σ_- (note that J_- is the primitive generator in this deformation), whereas the standard one contains functions of $\bar{\sigma}_3$ (J_3 is now the primitive one) and the geometrical meaning of both coordinates is completely different.

6. Concluding remarks

The systems presented here can be seen as basic examples of the implementation of integrable nonlinear interactions through quantum algebras. We would like to recall again the universality of this construction, which ensures the obtention of deformed integrable systems by using non-trivial representations of any quantum algebra with a Casimir element.

It is interesting to stress that co-algebra symmetries are also relevant at the undeformed level, since they account for the integrability properties of known systems like the isotropic N-dimensional oscillator and the Gaudin magnet. Note that these two systems are superintegrable and we have seen that, in the first case, some choices of the deformed Hamiltonian do preserve superintegrability. The question concerning the precise characterization of the deformations and dynamical Hamiltonians that follow this rule is an open question.

Finally, although further analysis of the deformed dynamics of these models is needed, two main features can be already be pointed out: the long-range nature of the interactions and their dependence on momenta. The latter fact can be explored through the generalized nonlinear oscillators (3.7) that appear as the one-particle Hamiltonians $H_z^{(1)}$ and from which the higher-dimensional systems are constructed. In this respect it is interesting to recall that some relations between generalized nonlinear oscillators and dissipation can be established [15, 16]. In the same direction, some connections between quantum algebras and this kind of phenomena have already been envisaged [17]. Finally, notice that long-range interactions in discrete systems can be linked to dispersive effects in the continuum limit (see [18] and references therein).

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